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On the semi-continuity of generalized inverses in Banach algebras ☆

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Abstract

The main purpose of this paper is to study the continuity of several kinds of generalized inverses of elements in a Banach algebra with identity. We first obtain a sufficient and necessary condition for the lower semi-continuity of reflexive generalized inverses as set-valued mappings. Based on this result, we characterize the continuity of the Moore–Penrose inverse in a C^*-algebra and therefore, derive some new and well-known criteria in operator theory.

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1. Introduction and preliminaries

Let $\Im$ be a Banach algebra with identity element e. We call an element $a \in \Im$ (*von Neumann*) *regular* if $a \in a\Im a$, give it a *generalized inverse* $b \in \Im$ for which $a = aba$. If

$$a = aba \quad \text{and} \quad b = bab,$$

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then b is called a *reflexive generalized inverse* of a. We always write a^+ for a reflexive generalized inverse of a. If $\Im$ is a C^*-algebra. An element $a^\dagger \in \Im$ is called the Moore–Penrose inverse of $a \in \Im$ if

$$a = aa^\dagger a, \quad a^\dagger = a^\dagger aa^\dagger, \quad (aa^\dagger)^* = aa^\dagger, \quad (a^\dagger a)^* = a^\dagger a.$$

Recall that every regular element has the Moore–Penrose inverse and the Moore–Penrose inverse is uniquely determined if it exists. However, the reflexive generalized inverse is not unique in general. This leads us to investigate the semi-continuity of reflexive generalized inverses as set-valued mapping. It seems to be the first time that the semi-continuity of reflexive generalized inverses as a set-valued mapping is studied. In Section 2, utilizing the method developed in [8,11], we obtain a sufficient and necessary condition for the lower semi-continuity of reflexive generalized inverses as set-valued mapping in a Banach algebra. Using this result, we study the continuity of the Moore–Penrose inverses which has been studied in [6,7,9,13,14]. We characterize the continuity of the Moore–Penrose inverse in a C^*-algebra and derive continuity criteria which unify and generalize the well-known criteria from operator theory [1–5,8,10–12,15].

2. Lower semi-continuity of the reflexive generalized inverse in Banach algebras

In this section, $\Im$ denotes a Banach algebra with identity element e. To prove our main results, we need the following lemmas. Unless other specified, letters a, b and c denote elements in $\Im$.

Lemma 2.1. *Let a^+ be a reflexive generalized inverse of a. Then aa^+ and a^+a are idempotents with $a\Im = aa^+\Im$ and $(e - a^+a)\Im = a^{-1}(0)$, where $a\Im = \{ab : b \in \Im\}$ and $a^{-1}(0) = \{x \in \Im : ax = 0\}$.*

Proof. It is easily checked. □

Lemma 2.2. *If $c = cac$. Then c is a reflexive generalized inverse of a if and only if $a\Im \cap c^{-1}(0) = \{0\}$.*

Proof. If c is a reflexive generalized inverse of a. Take any $y \in a\Im \cap c^{-1}(0)$, then $y = ax$ and $cax = cy = 0$ where $x \in \Im$. Hence, $y = ax = acax = 0$. This implies $a\Im \cap c^{-1}(0) = \{0\}$. Conversely, if $a\Im \cap c^{-1}(0) = \{0\}$. Note that $a - aca = a(e - ca) \in a\Im$ and $a - aca \in c^{-1}(0)$ since $c(a - aca) = ca - caca = 0$, we conclude that $a = aca$. The proof is complete. □

Lemma 2.3. *If a^+ is a reflexive generalized inverse of a and $\|b - a\| \cdot \|a^+\| < 1$. Then*

$$s = [e + a^+(b - a)]^{-1}a^+ = a^+[e + (b - a)a^+]^{-1}$$

satisfies $s = sbs$, $s\Im = a^+\Im$ and $s^{-1}(0) = (a^+)^{-1}(0)$.

Proof. Since $\|b - a\| \cdot \|a^+\| < 1$, $e + a^+(b - a)$ and $e + (b - a)a^+$ are invertible and a direct computation can give $[e + a^+(b - a)]^{-1}a^+ = a^+[e + (b - a)a^+]^{-1}$. Put $s = [e + a^+(b - a)]^{-1}a^+ = a^+[e + (b - a)a^+]^{-1}$. Obviously, $s\Im = a^+\Im$ and $s^{-1}(0) = (a^+)^{-1}(0)$. Let $d = e + a^+(b - a)$. Then $a^+ad = a^+a[e + a^+(b - a)] = a^+a + a^+aa^+b - a^+aa^+a = a^+b$ which implies $a^+bd^{-1} = a^+a$. Hence, $sbs = d^{-1}a^+bd^{-1}a^+ = d^{-1}a^+aa^+ = d^{-1}a^+ = s$. The proof is complete. □

Lemma 2.4. *If a^+ is a reflexive generalized inverse of a and $\|b-a\| \cdot \|a^+\| < 1$. Then $s = [e + a^+(b-a)]^{-1}a^+$ is a reflexive generalized inverse of b if and only if $b\Im \cap (a^+)^{-1}(0) = \{0\}$.*

Proof. It follows from Lemmas 2.2 and 2.3. □

Lemma 2.5. *If p and q are idempotents in $\Im$ with $\|p-q\| < 1$. Then $pq\Im = p\Im$.*

Proof. It suffices to show $p\Im \subset pq\Im$. Note $\|p-q\| < 1$, so $e-(p-q)$ is invertible. Hence, for any $x \in \Im$, $px = p[e-(p-q)][e-(p-q)]^{-1}x = pq[e-(p-q)]^{-1}x \in pq\Im$. The proof is complete. □

Theorem 2.1. *If a^+ is a reflexive generalized inverse of a and $a_n \to a$ in $\Im$. Then the following conditions are equivalent*:

(1) *for n large enough, a_n has a reflexive generalized inverse a_n^+ with*

$$a_n^+ \to a^+;$$

(2) $a_n\Im \cap (a^+)^{-1}(0) = \{0\}$ *for n large enough.*

Proof. (2) $\Rightarrow$ (1) Since $a_n \to a$, there exists an $N > 0$ such that for all $n \geqslant N$, $\|a_n - a\| \cdot \|a^+\| < 1$. By Lemma 2.4, $s_n = [e + a^+(a_n - a)]^{-1}a^+$ is a reflexive generalized inverse of a_n for $n \geqslant N$ and it is clear that $s_n \to a^+$.

(1) $\Rightarrow$ (2) If a_n has a reflexive generalized inverse a_n^+ such that $a_n^+ \to a^+$. Then $a_n^+a_n \to a^+a$. Hence, for n large enough,

$$\|a_n^+a_n - a^+a\| < 1 \quad \text{and} \quad \|a_n - a\| \cdot \|a^+\| < 1.$$

Since $e - a_n^+a_n$ and $e - a^+a$ are idempotents and $\|(e - a_n^+a_n) - (e - a^+a)\| < 1$, from Lemma 2.5, we have

$$(e - a^+a)(e - a_n^+a_n)\Im = (e - a^+a)\Im.$$

Hence, by Lemma 2.1,

$$(e - a^+a)a_n^{-1}(0) = a^{-1}(0).$$

Now, we show $a_n\Im \cap (a^+)^{-1}(0) = \{0\}$. If $y \in a_n\Im \cap (a^+)^{-1}(0)$, then $y = a_nx$ and $a^+a_nx = a^+y = 0$ where $x \in \Im$. Hence, $a[e + a^+(a_n - a)]x = ax + aa^+(a_n - a)x = aa^+a_nx = 0$ which means that $[e + a^+(a_n - a)]x \in a^{-1}(0) = (e - a^+a)a_n^{-1}(0)$. Thus, there exists a $z \in a_n^{-1}(0)$ such that

$$[e + a^+(a_n - a)]x = (e - a^+a)z = [e + a^+(a_n - a)]z.$$

Note $\|a_n - a\| \cdot \|a^+\| < 1$, so $x = z \in a_n^{-1}(0)$. Therefore, $y = a_nx = 0$. The proof is complete. □

Remark 2.1. In the linear operator case, Theorem 2.1 has been proved in [2,11].

Corollary 2.1. *Let a^+ be a reflexive generalized inverse of a and $a_n \to a$ in $\Im$. If $a_n\Im \subseteq a\Im$ or $a^{-1}(0) \subseteq a_n^{-1}(0)$, then for n large enough, a_n has a reflexive generalized inverse a_n^+ with $a_n^+ \to a^+$.*

Proof. By Theorem 2.1, it suffices to show that for n large enough, $a_n\Im \cap (a^+)^{-1}(0) = \{0\}$. First, if $a_n\Im \subseteq a\Im$. From Lemma 2.2, $a\Im \cap (a^+)^{-1}(0) = \{0\}$. Then $a_n\Im \cap (a^+)^{-1}(0) = \{0\}$. Next, if $a^{-1}(0) \subseteq a_n^{-1}(0)$, then $(e - a^+a)a_n^{-1}(0) = a^{-1}(0)$. As in the proof of Theorem 2.1, we can get what we desire. $\square$

Lemma 2.6. *If a^+ and $a^\oplus$ are two reflexive generalized inverses of a. Then $a^\sharp = a^+aa^\oplus$ is also a reflexive generalized inverse of a with $a^\sharp\Im = a^+\Im$ and $(a^\sharp)^{-1}(0) = (a^\oplus)^{-1}(0)$.*

Proof. It is easy to verify that $a^\sharp$ is a reflexive generalized inverse of a. Hence, by Lemma 2.1,

$$a^\sharp\Im = a^\sharp a\Im = a^+aa^\oplus a\Im = a^+a\Im = a^+\Im$$

and

$$(a^\sharp)^{-1}(0) = (e - aa^\sharp)\Im = (e - aa^+aa^\oplus)\Im = (e - aa^\oplus)\Im = (a^\oplus)^{-1}(0).$$

The proof is complete. $\square$

Theorem 2.2. *If a^+ and $a^\oplus$ are two reflexive generalized inverses of a and $a_n \to a$ in $\Im$. Then there exists an $N > 0$ such that for $n \geqslant N$,*

$$a_n\Im \cap (a^\oplus)^{-1}(0) = \{0\}$$

if and only if

$$a_n\Im \cap (a^+)^{-1}(0) = \{0\}.$$

Proof. Set $a^\sharp = a^+aa^\oplus$, then $a^\sharp$ is a reflexive generalized inverse of a with $a^\sharp\Im = a^+\Im$ and $(a^\sharp)^{-1}(0) = (a^\oplus)^{-1}(0)$. Put

$$\delta = (1 + \|a^+\| + \|a^\oplus\| + \|a^\oplus\|\|a^+\| + \|a^\oplus\|\|a\|\|a^+\|)^{-1}.$$

Then there exists an $N > 0$ such that for $n \geqslant N$, $\|a_n - a\| < \delta$. If

$$a_n\Im \cap (a^\oplus)^{-1}(0) = \{0\}$$

for all $n \geqslant N$. Hence, $a_n\Im \cap (a^\sharp)^{-1}(0) = \{0\}$. From Lemma 2.4, a_n has a reflexive generalized inverse $a_n^\sharp$ with $a_n^\sharp\Im = a^\sharp\Im$. Therefore,

$$a_n\Im = a_n a_n^\sharp\Im = a_n a^\sharp\Im = a_n a^+\Im. \tag{2.1}$$

In the following, we shall show $a_n\Im \cap (a^+)^{-1}(0) = \{0\}$. As in the proof of Theorem 2.1, we only need to prove

$$(e - a^+a)a_n^{-1}(0) = a^{-1}(0).$$

If $x \in a^{-1}(0)$, then by Eq. (2.1), there is a $y \in \Im$ such that $a_nx = a_na^+y$, i.e., $x - a^+y \in a_n^{-1}(0)$. Hence, $(e - a^+a)(x - a^+y) = x - a^+ax - a^+y + a^+aa^+y = x$. This implies that $a^{-1}(0) \subseteq (e - a^+a)a_n^{-1}(0)$. Clearly, the reverse inclusion holds. Thus, we have shown $(e - a^+a)a_n^{-1}(0) = a^{-1}(0)$. Therefore, $a_n\Im \cap (a^+)^{-1}(0) = \{0\}$.

Conversely, if $a_n\Im \cap (a^+)^{-1}(0) = \{0\}$ for $n \geqslant N$. Set $a^\natural = a^\oplus aa^+$. Just as the above discussions, we can prove $a_n\Im \cap (a^\oplus)^{-1}(0) = \{0\}$ for $n \geqslant N$. The proof is complete. $\square$

In the following, we shall give a sufficient and necessary condition for the lower semi-continuity of reflexive generalized inverses as set-valued mapping in a Banach algebra. For completeness, we introduce the definition of the lower semi-continuity of the set-valued mapping.

Definition 2.1 [16]. Let X be a Banach space and $g : X \longrightarrow 2^X$ be a set-valued map. The map g is called *lower semi-continuous* at the point $x \in X$ if and only if for every neighborhood $v(y)$ of every $y \in g(x)$, there exists a neighborhood $u(x)$ such that

$$g(u) \cap v(y) \neq \emptyset \quad \text{for all } u \in u(x).$$

The above definition can be phrased as follows: given any sequence $x_n \in X$ with $x_n \to x$, $g(x) \subseteq \underline{\lim} g(x_n) := \{y \in Y, \text{ there exists } y_n \in g(x_n) \text{ such that } y_n \to y\}$.

Theorem 2.3. *Let $\Im$ be a Banach algebra with identity element e and $a \in \Im$ have at least one reflexive generalized inverse. Then the following statements are equivalent*:

(1) *The set-valued map $g : a \to a^+$ is lower semi-continuous at a.*
(2) *a has a reflexive generalized inverse a^+ and there is a neighborhood $u(a)$ of a, such that for any $b \in u(a)$,*

$$b\Im \cap (a^+)^{-1}(0) = \{0\}.$$

(3) *a has a reflexive generalized inverse a^+ and given any sequence $a_n \in \Im$ with $a_n \to a$, there exists $N > 0$ such that for any $n \geqslant N$,*

$$a_n\Im \cap (a^+)^{-1}(0) = \{0\}.$$

Proof. It is obvious to see that (2) $\Leftrightarrow$ (3). We only shall show (1) $\Leftrightarrow$ (3). If $g : a \to a^+$ is lower semi-continuous at a, then for any $a^+ \in g(a)$ and any sequence $a_n \in \Im$ with $a_n \to a$, a_n has reflexive generalized inverse a_n^+ such that $a_n^+ \to a^+$. Hence, by Theorem 2.1, there exists $N > 0$ such that for any $n \geqslant N$, $a_n\Im \cap (a^+)^{-1}(0) = \{0\}$. Conversely, if (3) holds, then for any reflexive generalized inverse $a^\oplus$ of a and any sequence $a_n \in \Im$ with $a_n \to a$, by the statement (3), we have that there exists $N > 0$ such that for any $n \geqslant N$, $a_n\Im \cap (a^+)^{-1}(0) = \{0\}$. Thus, from Theorem 2.2, $a_n\Im \cap (a^\oplus)^{-1}(0) = \{0\}$ for n large enough. Using Theorem 2.1 again, we get that for n large enough, a_n has reflexive generalized inverse $a_n^\oplus$ such that $a_n^\oplus \to a^\oplus$ which means $a^\oplus \in \underline{\lim} g(a_n)$. This implies that $g(a) \subseteq \underline{\lim} g(a_n)$. The proof is complete. $\square$

Corollary 2.2. *Let $a \in \Im$ have at least one reflexive generalized inverse. If there is a neighborhood $u(a)$ of a such that for any $b \in u(a)$, $b\Im \subseteq a\Im$ or $a^{-1}(0) \subseteq b^{-1}(0)$, then the set-valued map $g : a \to a^+$ is lower semi-continuous at a.*

Remark 2.2. Even the Corollary 2.2 seems to be new even in operator theory.

3. Continuity of the Moore–Penrose inverse in C^*-algebras

In this section, let $\Im$ be a C^*-algebra with identity element e. Based on the results in Section 2, we characterize the continuity of the Moore–Penrose inverse in C^*-algebra.

Lemma 3.1 [5]. *If a has a generalized inverse a^+. Then a has a Moore–Penrose inverse $a^\dagger \in A$ defined by*

$$a^\dagger = p^* p[e - (p - p^*)^2]^{-1} a^+ q q^* [e - (q - q^*)^2]^{-1},$$

where $p = a^+ a$ and $q = aa^+$.

Theorem 3.1. *Let $a^\dagger$ be the Moore–Penrose inverse of a and $a_n \to a$ in $\Im$. Then the following conditions are equivalent*:

(1) *for n large enough, a_n has the Moore–Penrose inverse $a_n^\dagger$ with*
$$a_n^\dagger \to a^\dagger;$$
(2) $a_n \Im \cap (a^\dagger)^{-1}(0) = \{0\}$ *for n large enough.*

Proof. It is clear to see that (1) $\Rightarrow$ (2) comes from Theorem 2.1 since $a_n^\dagger$ is also a reflexive generalized inverse of a_n.

Now we show (2) $\Rightarrow$ (1). If for n large enough, $a_n \Im \cap (a^\dagger)^{-1}(0) = \{0\}$, then a_n has a reflexive generalized inverse a_n^+ such that $a_n^+ \to a^\dagger$. By Lemma 3.1, we get that for n large enough, a_n has the Moore–Penrose inverse $a_n^\dagger$ defined by

$$a_n^\dagger = p_n^* p_n [e - (p_n - p_n^*)^2]^{-1} a_n^+ q_n q_n^* [e - (q_n - q_n^*)^2]^{-1},$$

where $p_n = a_n^+ a_n$ and $q_n = a_n a_n^+$. Note $a_n^+ \to a^\dagger$, so $p_n = a_n^+ a_n \to a^\dagger a$ and $q_n = a_n a_n^+ \to aa^\dagger$. Hence, it is easy to see $a_n^\dagger \to a^\dagger$. The proof is complete. □

Remark 3.1. Theorem 3.1 has been proved in the case of linear operator in [2,8]. From Theorems 2.1, 2.2 and 3.1, we can get the following theorem.

Theorem 3.2. *Let $a^\dagger$ be the Moore–Penrose inverse of a and a^+ be a reflexive generalized inverse of a and $a_n \to a$ in $\Im$. Then the following conditions are equivalent*:

(1) *for n large enough, a_n has the Moore–Penrose inverse $a_n^\dagger$ with $a_n^\dagger \to a^\dagger$*;
(2) $a_n \Im \cap (a^\dagger)^{-1}(0) = \{0\}$ *for n large enough*;
(3) $a_n \Im \cap (a^+)^{-1}(0) = \{0\}$ *for n large enough*;
(4) *for n large enough, a_n has a reflexive generalized inverse a_n^+ with $a_n^+ \to a^+$.*

Remark 3.2. Theorem 3.2 seems to be new even in operator theory.

Corollary 3.1. *Let $a^\dagger$ be the Moore–Penrose inverse of a and $a_n \to a$ in $\Im$. If $a_n \Im \subseteq a\Im$ or $a^{-1}(0) \subseteq a_n^{-1}(0)$, then for n large enough, a_n has the Moore–Penrose inverse $a_n^\dagger$ with $a_n^\dagger \to a^\dagger$.*

Proof. The proof is similar to Corollary 2.1, we omit it. □

Corollary 3.2. *Let a^+ be a reflexive generalized inverse of a and $a_n \to a$ in $\Im$ and $a_n \Im \cap (a^+)^{-1}(0) = \{0\}$. Then for n large enough, a_n has the Moore–Penrose inverse $a_n^\dagger$ defined by*

$$a_n^\dagger = p_n^* p_n [e - (p_n - p_n^*)^2]^{-1} [e + a^+(a_n - a)]^{-1} a^+ q_n q_n^* [e - (q_n - q_n^*)^2]^{-1}$$

such that $a_n^\dagger$ converges to the Moore–Penrose inverse of a, where $p_n = [e + a^+(a_n - a)]^{-1} a^+ a_n$, $q_n = a_n [e + a^+(a_n - a)]^{-1} a^+$.

Proof. It comes from Lemmas 2.4, 3.1 and Theorem 3.2. $\square$

4. The linear operator case

Now we are going to specialize the results of the preceding sections to the case of linear operators. We need a lemma which plays a crucial role in such specialization. We denote by $R(T)$ and $N(T)$ the range and the kernel of the linear operator T.

Lemma 4.1. *Let X be a Banach space and $\Im$ denote the Banach algebra of all bounded linear operators on X. Let $T_0, T \in \Im$ and T_0^+ be a reflexive generalized inverse of T_0. Then the following conditions are equivalent*:

(1) $T\Im \cap (T_0^+)^{-1}(0) = \{0\}$;
(2) $R(T) \cap N(T_0^+) = \{0\}$.

We remark that the 0 in statement (1) denotes the zero operator on X (i.e., zero element in $\Im$) and the 0 in statement (2) is the zero element of X.

Proof. (1) $\Rightarrow$ (2) If $y \in R(T) \cap N(T_0^+)$, then there exists $x \in X$ such that $y = Tx$ and $T_0^+Tx = T_0^+y = 0$. Since span$\{x\}$ is a 1-dimensional subspace of X, X has the topological decomposition: $X = \text{span}\{x\} \oplus M$, where M is a closed subspace of X. For any $z \in X$, $z = z_1 + z_2$, where $z_1 \in \text{span}\{x\}$ and $z_2 \in M$, we define the linear operator S on X as follows: $Sz = z_1$. Hence, for all $z \in X$, $T_0^+TSz = 0$, i.e., $T_0^+TS \equiv 0$. This implies that $TS \in T\Im \cap (T_0^+)^{-1}(0)$. Thus, by statement (1), $TS = 0$. Therefore, $y = Tx = TSx = 0$.

(2) $\Rightarrow$ (1) If $R \in T\Im \cap (T_0^+)^{-1}(0)$, then there exists $S \in \Im$ such that $R = TS$ and $T_0^+TS = T_0^+R = 0$. Hence, for any $x \in X$, $T_0^+TSx = 0$ which means that $TSx \in R(T) \cap N(T_0^+)$. Thus, by statement (2), $TSx = 0$, i.e., $Rx = 0$. We conclude that $R = 0$ since x is arbitrary in X. The proof is complete. $\square$

Now we specialize our results in the preceding sections to the operator case. For simplicity, we do it only in the case of the linear operators on Hilbert space and give some results which seem to be new in operator theory. In the following, we always suppose that $\Im$ denotes the C^*-algebra of all linear operators on a Hilbert space H and T_0, T_n are in $\Im$ with $T_n \to T_0$.

Theorem 4.1. *Let T_0 has at least one reflexive generalized inverse. Then the following statements are equivalent*:

(1) *The set-valued mapping $G : T \to T^+$ is lower semi-continuous at T_0*;
(2) *T_0 has a reflexive generalized inverse T_0^+ and there is a neighborhood $U(T_0)$ of T_0, such that for any $T \in U(T_0)$,*

$$R(T) \cap N(T_0^+) = \{0\}.$$

Theorem 4.2. *Let $T_0^\dagger$ and T_0^+ be the Moore–Penrose inverse and a reflexive generalized inverse of T_0, respectively. Then the following statements are equivalent*:

(1) *for n large enough*, T_n *has the Moore–Penrose inverse* $T_n^\dagger$ *with* $T_n^\dagger \to T_0^\dagger$;
(2) $R(T_n) \cap N(T^\dagger) = \{0\}$ *for n large enough*;
(3) $R(T_n) \cap N(T_0^+) = \{0\}$ *for n large enough*;
(4) *for n large enough*, T_n *has a reflexive generalized inverse* T_n^+ *with* $T_n^+ \to T_0^+$.

Corollary 4.1. *Let* T_0^+ *be a reflexive generalized inverse of* T_0 *and* $R(T_n) \cap N(T_0^+) = \{0\}$. *Then for n large enough*, T_n *has the Moore–Penrose inverse* $T_n^\dagger$ *defined by*

$$T_n^\dagger = P_n^* P_n[I - (P_n - P_n^*)^2]^{-1}[I + T_0^+(T_n - T_0)]^{-1} T_0^+ Q_n Q_n^*[I - (Q_n - Q_n^*)^2]^{-1}$$

such that $T_n^\dagger$ *converges to the Moore–Penrose inverse* $T_0^\dagger$ *of* T_0, *where* I *denotes the identity operator on Hilbert space* H, $P_n = [I + T_0^+(T_n - T_0)]^{-1} T_0^+ T_n$ *and* $Q_n = T_n[I + T_0^+(T_n - T_0)]^{-1} T_0^+$.

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